

# ALMOST SIMPLE GROUPS WITH SOCLE $L_n(q)$ ACTING ON STEINER QUADRUPLE SYSTEMS

MICHAEL HUBER

**ABSTRACT.** Let  $N = L_n(q)$ ,  $n \geq 2$ ,  $q$  a prime power, be a projective linear simple group. We classify all Steiner quadruple systems admitting a group  $G$  with  $N \leq G \leq \text{Aut}(N)$ . In particular, we show that  $G$  cannot act as a group of automorphisms on any Steiner quadruple system for  $n > 2$ .

## 1. INTRODUCTION

All Steiner quadruple systems which admit a flag-transitive group of automorphisms were classified in [8]. The most interesting examples which occur have an almost simple group with socle  $L_2(q)$  as group of automorphisms. In this note, we examine for these type of groups the general case when flag-transitivity is omitted. Before stating the main result, we present the examples that arise in our consideration.

**Example 1.**  $\mathcal{D}$  is a Steiner quadruple system  $\text{SQS}(3^d + 1)$  whose points are the elements of the projective line  $\mathbb{F}_{3^d} \cup \{\infty\}$  and whose blocks are the images of  $\mathbb{F}_3 \cup \{\infty\}$  under  $PGL_2(3^d)$  with  $d \geq 2$  (resp.  $L_2(3^d)$  with  $d > 1$  odd), and  $L_2(3^d) \leq G \leq P\Gamma L_2(3^d)$ . The derived design at any given point is the Steiner triple system  $\text{STS}(3^d)$  whose points and blocks are the points and lines of the affine space  $AG(d, 3)$ . In this case,  $G$  acts flag-transitively on  $\mathcal{D}$ .

**Example 2.**  $\mathcal{D}$  is a Steiner quadruple system  $\text{SQS}(q + 1)$  whose points are the elements of  $\mathbb{F}_q \cup \{\infty\}$  with a prime power  $q \equiv 7 \pmod{12}$  and whose blocks are the images of  $\{0, 1, \infty, \varepsilon\}$  under  $L_2(q)$ , where  $\varepsilon$  is a primitive sixth root of unity in  $\mathbb{F}_q$ , and  $L_2(q) \leq G \leq P\Gamma L_2(q)$ . The derived design at any given point is the *Netto triple system*  $N(q)$ , a detailed description of which can be found in [4, Section 3]. Here,  $G$  acts flag-transitively on  $\mathcal{D}$ .

**Example 3.**  $\mathcal{D}$  is a Steiner quadruple system  $\text{SQS}(3^{2d} + 1)$  whose points are the elements of  $\mathbb{F}_{3^{2d}} \cup \{\infty\}$  and whose blocks are the disjoint union

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of the images of  $\{0, 1, -1, \infty\}$  and  $\{0, 1, a, \infty\}$  under  $L_2(3^{2d})$  with  $d \geq 1$ ,  $a \notin (\mathbb{F}_{3^{2d}}^*)^2$ , and  $L_2(3^{2d}) \leq G \leq P\Gamma L_2(3^{2d})$ . In this case,  $G$  has two orbits on the 3-subsets and hence on the blocks of  $\mathcal{D}$ . Therefore, flag-transitivity cannot hold.

Our main result is as follows.

**Main Theorem.** *Let  $\mathcal{D}$  be a non-trivial Steiner quadruple system  $SQS(v)$  of order  $v$ , and  $N \leq G \leq \text{Aut}(N)$  with a projective linear simple group  $N = L_n(q)$ ,  $n \geq 2$ ,  $q$  a prime power,  $v = \frac{q^n-1}{q-1}$ . Then  $G \leq \text{Aut}(\mathcal{D})$  acts on  $\mathcal{D}$  if and only if one of the cases described in Examples 1, 2, 3 above occurs (up to isomorphism).*

## 2. PRELIMINARIES

For positive integers  $t \leq k \leq v$  and  $\lambda$ , we define a  $t$ -( $v, k, \lambda$ ) *design* to be a finite incidence structure  $\mathcal{D} = (X, \mathcal{B}, I)$ , where  $X$  denotes a set of *points*,  $|X| = v$ , and  $\mathcal{B}$  a set of *blocks*,  $|\mathcal{B}| = b$ , satisfying the following properties: (i) each block  $B \in \mathcal{B}$  is incident with  $k$  points; and (ii) each  $t$ -subset of  $X$  is incident with  $\lambda$  blocks. A *flag* is an incident point-block pair.

For historical reasons, a  $t$ -( $v, k, 1$ ) design is called a *Steiner  $t$ -design* or a *Steiner system*. We note that in this case each block is determined by the set of points which are incident with it, and thus can be identified with a  $k$ -subset of  $X$  in a unique way. A *Steiner triple system* of order  $v$  is a 2-( $v, 3, 1$ ) design. A *Steiner quadruple system* of order  $v$  is a 3-( $v, 4, 1$ ) design, and will be denoted in the following by  $SQS(v)$ . The case  $v = 4$  yields the *trivial*  $SQS(v)$ . A simple example of a Steiner quadruple system is the  $SQS(2^n)$  consisting of the points and planes of the  $n$ -dimensional binary affine space  $AG(n, 2)$  for each  $n \geq 2$ . Using recursive constructions, H. Hanani [6] showed that the following condition for the existence of a  $SQS(v)$  (the necessity of which is easy to see) is also sufficient:

**Proposition 4.** (Hanani, 1960). *A Steiner quadruple system  $SQS(v)$  of order  $v$  exist if and only if*

$$v \equiv 2 \text{ or } 4 \pmod{6} \quad (v \geq 4).$$

For  $v = 8$  and  $v = 10$  there exists a  $SQS(v)$  in each case, unique up to isomorphism. These are the affine space  $AG(3, 2)$  and the Möbius plane of order 3, see Barrau [1], 1908. For  $v = 14$  we have exactly four distinct isomorphism types, cf. Mendelsohn & Hung [9], 1972. For  $v = 16$  there are exactly 1,054,163 distinct isomorphism types, see Kaski, Östergård & Pottonen [10], 2006.

If  $\mathcal{D} = (X, \mathcal{B}, I)$  is a  $t$ -( $v, k, \lambda$ ) design with  $t \geq 2$ , and  $x \in X$  arbitrary, then the *derived* design with respect to  $x$  is  $\mathcal{D}_x = (X_x, \mathcal{B}_x, I_x)$ , where  $X_x = X \setminus \{x\}$ ,  $\mathcal{B}_x = \{B \in \mathcal{B} : (x, B) \in I\}$  and  $I_x = I|_{X_x \times \mathcal{B}_x}$ . In this case,  $\mathcal{D}$  is also called an *extension* of  $\mathcal{D}_x$ . Obviously,  $\mathcal{D}_x$  is a  $(t-1)$ -( $v-1, k-1, \lambda$ ) design.

For a group  $G \leq \text{Aut}(\mathcal{D})$  of automorphisms of  $\mathcal{D}$ , let  $G_B$  denote the set-wise stabilizer of a block  $B \in \mathcal{B}$ . All other notation is standard.

A detailed account on Steiner systems can be found, e.g., in [2] and [3]. Comprehensive survey articles on Steiner quadruple systems include [7] and [11].

### 3. PROOF OF THE MAIN THEOREM

Let  $\mathcal{D}$  be a non-trivial Steiner quadruple system SQS( $v$ ) of order  $v$ , and  $N \leq G \leq \text{Aut}(N)$  with a projective linear simple group  $N = L_n(q)$ ,  $n \geq 2$ ,  $q$  a prime power. Here,  $(n, q) \neq (2, 2), (2, 3)$ . We consider the natural action of  $G$  on the projective space  $PG(n-1, q)$ ,  $v = \frac{q^n-1}{q-1}$  (note that  $L_n(q)$  has two doubly transitive permutation representations of the given degree if  $n > 2$ ). We remark that besides the fact that  $N = L_n(q)$  is not simple for  $(n, q) = (2, 2), (2, 3)$ , it is obviously not possible to obtain in these cases a non-trivial SQS( $v$ ) by definition.

**Lemma 5.** *Let  $N = L_2(q)$ ,  $v = q+1$ . If  $N \leq G \leq \text{Aut}(N)$  is 3-homogeneous, then the cases as in Examples 1 and 2 hold in the Main Theorem.*

*Proof.* If  $G$  is 3-homogeneous, then in particular  $G$  is flag-transitive. Hence, we may argue as in the corresponding case in [8] to obtain the known classes of Steiner quadruple systems. We note that in doing so, we only need to rely on [4], not on the classification of the finite simple groups.  $\square$

**Lemma 6.** *Let  $N = L_2(q)$ ,  $v = q+1$ . If  $N \leq G \leq \text{Aut}(N)$  is not 3-homogeneous, then the case described in Example 3 holds in the Main Theorem.*

*Proof.* If we assume that  $G$  is not 3-homogeneous, which is the case if and only if  $q \equiv 1 \pmod{4}$ , then  $G$  has more than one orbit on the blocks and hence there cannot exist any flag-transitive SQS( $v$ ). However, we will show that there exists a class of Steiner quadruple systems on which  $G$  operates point 2-transitively. Since  $N$  is a 2-transitive permutation group, we may restrict ourselves to the case  $N = G$ . As  $PGL_2(q)$  is 3-homogeneous, the unique orbit under  $PGL_2(q)$  on the 3-subsets splits under  $G$  in exactly two orbits of equal length. By the definition of Steiner quadruple systems, it follows that  $G$  has exactly two orbits on the blocks. These have equal length as for any block  $B \in \mathcal{B}$  in each orbit, the representation  $G_B \rightarrow \text{Sym}(B) \cong S_4$  is faithful and thus

$$G_B \cong S_4.$$

We remark that  $G_B$  has then four Sylow 3-subgroups. By Proposition 4 and the fact that  $q \equiv 1 \pmod{4}$ , we have to distinguish the following two cases:

Case (a):  $q = 3^{2d}$ ,  $d \geq 1$ .

Since  $3 \mid q$ , each Sylow 3-subgroup has exactly one fixed point. Thus, we have at most one orbit of length 4 under  $G_B$ . On the other hand, the normalizer of a Sylow 3-subgroup in the symmetric group  $S_3$  is  $S_3$  itself,

hence  $S_3$  fixes the respective fixed point. The stabilizer of that fixed point in  $S_4$  has order at least 6. But, as it is 3-closed, it cannot be  $S_4$  itself. Moreover, it cannot be the alternating group  $A_4$  because the latter does not contain  $S_3$ . Thus, it can only have order 6. Therefore, there exists at least one orbit of length 4. Hence, we have in each of the two orbits on the blocks exactly one orbit of length 4 under  $G_B$ . This yields the circle geometries described in Example 3, where we choose  $a \notin (\mathbb{F}_q^*)^2$ , since in general  $-1 \in (\mathbb{F}_q^*)^2 \Leftrightarrow q \equiv 1 \pmod{4}$ . As  $2^4 \mid (3^d - 1)(3^d + 1)(3^{2d} + 1) = 3^{4d} - 1$  for all  $d \geq 1$ , we conclude that for  $q = 3^{2d}$ ,  $d \geq 1$ , always  $q^2 \equiv 1 \pmod{16}$  holds. Thus, we have in  $G$

$$\frac{(q+1)q(q-1)}{24}$$

many subgroups isomorphic to  $S_4$  on two conjugacy classes of equal length (cf. [5, p. 285]). As we have precisely

$$\frac{b}{2} = \frac{(q+1)q(q-1)}{2 \cdot 24}$$

circles on each orbit of blocks, we obtain no further  $\text{SQS}(v)$ .

Case (b):  $q \equiv 1 \pmod{12}$ .

Let us assume that we have an orbit of length 4 under  $G_B$ . Then, the stabilizer, say  $U$ , of a point in  $G_B$  is isomorphic to  $S_3$ . In  $U$ , we have a normal subgroup of order 3, which has exactly two fixed points as in particular  $3 \mid q-1$ . But, as these are left fixed by an involution in  $U$ , clearly  $U$  has two fixed points. On the other hand, the stabilizer on two points in  $L_2(q)$  is cyclic, which leads to a contradiction as  $S_3$  is non-Abelian. Hence, there cannot exist any  $\text{SQS}(v)$  in this case.  $\square$

**Lemma 7.** *Let  $N = L_n(q)$ ,  $n \geq 3$ ,  $v = \frac{q^n-1}{q-1}$ . Then  $N \leq G \leq \text{Aut}(N)$  cannot act as a group of automorphisms on any  $\text{SQS}(v)$ .*

*Proof.* Here  $\text{Aut}(N) = P\Gamma L_n(q) \rtimes \langle \iota_\beta \rangle$ , where  $\iota_\beta$  denotes the graph automorphism induced by the inverse-transpose map  $\beta : GL_n(q) \rightarrow GL_n(q)$ ,  $x \mapsto {}^t(x^{-1})$ . If  $n = 3$ , then  $v = q^2 + q + 1$  is always odd, in contrast to Proposition 4. For  $n > 3$ , we establish the claim via induction over  $n$ . Let us assume that there is a counter-example with  $n$  minimal. Without restriction, we can choose three distinct points  $x, y, z$  from a hyperplane  $\mathcal{H}$  of  $PG(n-1, q)$ . The translation group  $T(\mathcal{H})$  acts regularly on the points of  $PG(n-1, q) \setminus \mathcal{H}$  and trivially on  $\mathcal{H}$ . Hence, the unique block  $B \in \mathcal{B}$  which is incident with the 3-subset  $\{x, y, z\}$  must be contained completely in  $\mathcal{H}$ , since otherwise it would contain all points of  $PG(n-1, q) \setminus \mathcal{H}$ , yielding a contradiction. Thus,  $\mathcal{H}$  induces a  $\text{SQS}(\frac{q^{n-1}-1}{q-1})$  on which  $G$  with socle  $L_{n-1}(q)$  operates. Inductively, we obtain the minimal counter-example for  $n = 3$ , which we know is impossible. This verifies the claim.  $\square$

**Proof of Main Theorem:** The result is obtained by putting together Lemmas 5, 6, and 7.

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WILHELM SCHICKARD INSTITUTE FOR COMPUTER SCIENCE, UNIVERSITY OF TUEBINGEN, SAND 13, D-72076 TUEBINGEN, GERMANY

*E-mail address:* michael.huber@uni-tuebingen.de